ANOMALY, CHARGE QUANTIZATION AND FAMILY

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We first review the three known chiral anomalies in four dimensions and then use the anomaly free conditions to study the uniqueness of quark and lepton representations and charge quantizations in the standard model. We also extend our results to theory with an arbitrary number of color. Finally, we discuss the family problem.

Although the standard model [1] of $SU(3)_C \times SU(2)_L \times U(1)_Y$ has been remarkably successful experimentally, there are several puzzles, such as why the electric charges of quarks and leptons are quantized and why there are three fermion families? In this talk I would like to study these two puzzles in the viewpoint of the chiral gauge anomaly cancellations.

It is well-known that the anomaly free conditions arising from the theoretical requirements of renormalizability and self-consistency are the most elegant tool to test the gauge theory. Three anomalies thus far have been identified for chiral gauge theories in four dimensions: (1) The triangular (perturbative) chiral gauge anomaly,[2] which must be canceled to avoid the breakdown of gauge invariance and renormalizability of the theory; we call this the *triangular* anomaly. (2) The global (non-perturbative) $SU(2)$ chiral gauge anomaly,[3] which must be absent in order to define the fermion integral in a gauge invariant way; we call this the *global* anomaly. This anomaly was first pointed out by Witten,[3] and is known as the Witten $SU(2)$ anomaly. He showed in 1982 that any $SU(2)$ gauge theory with an odd number of left-handed fermion (Weyl) doublets is mathematically inconsistent. (3) The mixed (perturbative) chiral gauge-gravitational anomaly,[4,5] which must be canceled in order to ensure general covariance of the theory; we call this the *mixed* anomaly. This anomaly was first discussed by Delburgo and Salam[4] in 1972 and its consequences studied by Alvarez-Gaumé and Witten[5] in 1983, who concluded that a necessary condition for consistency of the theory coupled to gravity is that the sum of the $U(1)$ charges of the left-handed fermions vanishes, i.e., $TrQ = 0$.

We now review the three chiral anomalies for the simple Lie groups.

1. *The Triangular Anomaly*. It has been shown[6] that the simple Lie groups: $SU(2)$, $SO(2k+1)(k>2)$, $SO(4k)(k \geq 2)$, $SO(4k+2)(k \geq 2)$, $Sp(2k)$, G_2, F_4, E_6, E_7, and E_8 are safe groups. The only simple groups with possible triangular anomaly are the unitary groups $SU(n)(n \geq 3)$. Therefore, if we start with the groups which do not contain $SU(n)(n \geq 3)$ group, the

theory will be free of triangular anomaly.

2. *The Global Anomaly.* We classify the simple Lie groups G into the following two classes. (I) $Sp(2k)(Sp(2) \simeq SU(2))$. These groups[7] have the property of

$$\mathbf{\Pi}_4(Sp(2k)) = \mathbf{Z}_2 \ , \tag{1}$$

where $\mathbf{\Pi}_4$ is the fourth homotopy group and $\mathbf{Z}_2$ is the two-valued discrete group (like parity). According to Witten,[3] the group $G^{(I)} = Sp(2k)$ has global anomaly if the number of fermion zero modes (for $SU(2)$ group, it is equal to the number of fermion doublets) is odd. (II) $SU(n)(n \geq 3), SO(2k+1)(k > 2), SO(4k)(k \geq 2), SO(4k+2)(k \geq 2), G_2, F_4, E_6, E_7$, and E_8. These groups ($G^{(II)}$) have no global anomaly since their fourth homotopy groups are trivial,[3,7] i.e.,

$$\mathbf{\Pi}_4(G^{(II)}) = 0 \ . \tag{2}$$

However, the interesting question[8] arises as to how one can know at the level of $G^{(II)}$ whether such a theory is global anomaly-free when $G^{(II)}$ breaks down to groups which contain $G^{(I)}$. Recently, we present a sufficient condition[8] that for any simple group G, containing $Sp(2k)$ as a subgroup, and for which $\mathbf{\Pi}_4(G) = 0$, the vanishing of the triangular perturbative anomaly for Weyl representations of G will guarantee the absence of the global non-perturbative $Sp(2k)$ anomaly.

3. *The Mixed Anomaly.* This anomaly is non-trivial only for the theory in which there is $U(1)$ symmetry with non-zero total charges.[5] Obviously, all the simple Lie groups ($G^{(I),(II)}$) are safe groups. Furthermore, when these groups break down to groups which contain $U(1)$, e.g.,

$$G \to g \times \prod_i U(1)_i \ , \tag{3}$$

unlike the previous case, there is no mixed anomaly since the $U(1)$ operators are the generators of G and must be traceless.

The triangular anomaly-free of the standard model was first noted[9] in 1972 for each quark-lepton family. It was clear that with only the triangular anomaly-free condition[10] one could not explain the empirically determined quark-lepton representations and their quantized hypercharges. We now study[11] the question of the uniqueness of quarks and leptons in the standard model by insisting on all three anomaly-free conditions. With an arbitrary color number N (≥ 3), we begin by allowing an arbitrary number of (left-handed) Weyl representations under the group of $SU(N) \times SU(2) \times U(1)$, i.e.,

$$
\begin{array}{lll}
SU(N) \times SU(2) \times U(1) & & \\
\quad N & 2 & Q_i\,,\ \ i=1,\cdots,j \\
\quad N & 1 & Q'_i\,,\ \ i=1,\cdots,k \\
\quad \overline{N} & 1 & \overline{Q}_i\,,\ \ i=1,\cdots,l \\
\quad \overline{N} & 2 & \overline{Q}'_i\,,\ \ i=1,\cdots,m \\
\quad 1 & 2 & q_i\,,\ \ i=1,\cdots,n \\
\quad 1 & 1 & \overline{q}_i\,,\ \ i=1,\cdots,p
\end{array}
\tag{4}
$$

where the integers j, k, l, m, n and p and the $U(1)$ charges are all arbitrary. The triangular anomaly free conditions then lead to the following equations:

$$
\begin{aligned}
[SU(N)]^3 \ &: \ \sum_{i=1}^{j} 2 + \sum_{i=1}^{k} 2 - \sum_{i=1}^{l} 1 - \sum_{i=1}^{m} 2 = 0 \ , \\
[SU(N)]^2\, U(1) \ &: \ 2\sum_{i=1}^{j} Q_i + \sum_{i=1}^{k} Q'_i + \sum_{i=1}^{l} \overline{Q}'_i + 2\sum_{i=1}^{m} \overline{Q}'_i = 0 \ , \\
[SU(2)]^2\, U(1) \ &: \ N\sum_{i=1}^{j} Q_i + N\sum_{i=1}^{m} \overline{Q}'_i + \sum_{i=1}^{n} q_i = 0 \ , \\
[U(1)]^3 : N\sum_{i=1}^{j} Q_i^3 + \frac{N}{2}\sum_{i=1}^{k} Q_i'^3 &+ \frac{N}{2}\sum_{i=1}^{l} \overline{Q}_i^3 + N\sum_{i=1}^{m} \overline{Q}_i'^3 + \sum_{i=1}^{n} q_i^3 + \frac{1}{2}\sum_{i=1}^{p} \overline{q}_i^3 = 0 \ .
\end{aligned}
\tag{5}
$$

The global $SU(2)$ anomaly-free condition is

$$
N\,j + N\,m + n = E \ , \tag{6}
$$

where E is an even integer. Finally the mixed anomaly-free condition is

$$
[U(1)] : N\sum_{i=1}^{j} Q_i + \frac{N}{2}\sum_{i=1}^{k} Q'_i + \frac{N}{2}\sum_{i=1}^{l} \overline{Q}_i + N\sum_{i=1}^{m} \overline{Q}'_i + \sum_{i=1}^{n} q_i + \frac{1}{2}\sum_{i=1}^{p} \overline{q}_i = 0 \ . \tag{7}
$$

The requirements of minimality and the three anomaly-free conditions [Eqs. (5)-(7)] lead to the values: (I) if N= even #, $j = 1$, $k = 0$, $l = 2$, $m = n = p = 0$, and

$$
Q_1 = 0\,,\ \overline{Q}_1 = -\overline{Q}_2\,; \tag{8}
$$

and (II) if N= odd #, $j=1$, $k=0$, $l=2$, $m=0$, $n=1$, $p=1$, and to two solutions of $U(1)$ charges

$$Q_1 = \frac{1}{N}\,,\ \overline{Q}_1 = -\frac{N+1}{N}\,,\ \overline{Q}_2 = \frac{N-1}{N}\,,\ \overline{q}_1 = -2q_1 = -2,, \tag{9}$$

$$Q_1 = q_1 = \overline{q}_1 = 0\,,\ \overline{Q}_1 = -\overline{Q}_2\,, \tag{10}$$

where we have chosen the normalization $q_1 = -1$ in Eq. (9). For $N=3$, the solutions in Eqs. (9) and (10) are the "standard model" and the so called "bizarre" ones, respectively. We note that the "inert" state (1,1,0) for the "bizarre" solution is a non-chiral representation and it must be excluded. It is interesting to note that the "bizarre" solution may be viewed as the standard one when $N \to \infty$. Without considering the "bizarre" solution, for the odd number of color, all the $U(1)$ charges are uniquely determined. In this case, the resulting Weyl representations of $SU(N)$ and $SU(2)$ and their $U(1)$ charges are those in the standard model if N=3 (cf. Table 1). The electric charges of quarks and leptons for an arbitrary odd number of color N are given in Table 1 where the electroweak symmetry is spontaneously broken down to $U(1)_{EM}$ by the Higgs mechanism.

Table 1. The quantum numbers of quark and lepton representations under $SU(N)_C \times SU(2)_L \times U(1)_Y$ and $SU(N)_C \times U(1)_{EM}$

Particles $(i=1,2,3)$	$SU(N)_C \times SU(2)_L \times U(1)_Y$			$\to$	$SU(N)_C \times U(1)_{EM}$	
$\begin{pmatrix} u \\ d \end{pmatrix}_L^i$	N	2	$\frac{1}{N}$		$\begin{pmatrix} N \\ N \end{pmatrix}$	$\begin{pmatrix} \frac{N+1}{2N} \\ -\frac{N-1}{2N} \end{pmatrix}$
$u_L^{c\ i}$	$\overline{N}$	1	$-\frac{N+1}{N}$		$\overline{N}$	$-\frac{N+1}{2N}$
$d_L^{c\ i}$	$\overline{N}$	1	$\frac{N-1}{N}$		$\overline{N}$	$\frac{N-1}{2N}$
$\begin{pmatrix} \nu \\ e \end{pmatrix}_L^i$	1	2	-1		$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$e_L^{c\ i}$	1	1	2		1	1

For the standard model of $SU(3)_C \times SU(2)_L \times U(1)_Y$, we thus find that the requirements of minimality and freedom from all three chiral gauge anomalies lead to a unique set of Weyl representations (and their $U(1)_Y$ charges) of

the standard group that correspond to the observed quarks and leptons of one family. Furthermore, the $U(1)_Y$ charges of these quarks and leptons are quantized and correctly determined by adding the mixed anomaly-free condition and thus a long-standing puzzle of the electric charge quantization of quark and lepton can be solved within the content of the standard model.

In spite of the success of the standard model, it is still a mystery why the three anomaly cancellations, especially the global and the mixed ones, should be satisfied. Naturally one hopes that new physics beyond the standard model can provide us an explanation to this question. From the above studies we see that the three anomaly-free conditions in the standard model may be automatically satisfied if it comes from a large group, especially, a grand unification group. For example, with the E_6 grand unification theory, the triangular, the global, and the mixed anomalies are trivial at the level of E_6 which guarantees their freedom at the standard group level. We thus conclude that the resolution of the question of the uniqueness of the massless fermion representations and $U(1)_Y$ charges for the standard group – when viewed from the standpoint of the perturbative triangular and mixed chiral gauge-gravitational anomalies and the absence of the non-perturbative global $SU(2)$ chiral gauge anomaly in four dimensions – argues strongly for some new physics beyond the standard model.

Finally, we discuss the family issue. It is clear that, as one can see from the above study, the imposition of all three anomaly-free conditions for the standard model does not shed any immediate light on the "generation problem". In fact, the quantum numbers in Table 1 are generation blind. Moreover, if one enlarges the standard group to include an $SU(2)$ or $SU(3)$ group, one can show that the theories are precisely the one family fermion structure of the left-right symmetric model[11] $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)$ and the chiral-color model,[12] $SU(3)_{CL} \times SU(3)_{CR} \times SU(2)_L \times U(1)_Y$, respectively, instead of having a family group. Clearly, some new ideas[13] are needed to constrain on the number of families which would be a key to the new physics. We now present a toy model which gives rise to three families of quarks and leptons. In the standard model, in each family there are 15 Weyl spinors. With a right handed neutrino, the number becomes 16. For three families, the total numbers are 48. One may put all these 48 Weyl spinors into a flavor box to form a large global symmetry as $U(48)$.[13] From the study in Eqs. (4)-(10), we can extend the group of $SU(N) \times SU(2) \times U(1)$ with both even and odd numbers of N to a larger group of $SU(N) \times SU(2) \times SU(2)$ in which N has to be an even number as shown in Table 2. For N=4, it is just the Pati-Salam model,[14] which contains a right-handed neutrino. We remark that the representations under $SU(N) \times SU(2) \times SU(2)$ in Table 2 are unique unlike the case with a $U(1)$

symmetry and there is no more "bizarre" solution like the one in Eq. (10).

Table 2. The fermion quantum numbers under $SU(N) \times SU(2) \times SU(2)$

$SU(N)_C$	$\times$	$SU(2)$	$\times$	$SU(2)$
N		2		1
$\overline{N}$		1		2

We now take the global flavor symmetry $U(48)$ and gauge its subgroup $SU(12) \times SU(2) \times SU(2)$ so that the fermions transform according to the representations given in Table 2 with $N = 12$. Thus, the model is a generalized Pati-Salam theory with the color being 12. The symmetry breaking chains by various suitable scalars are given as follows:

$$
\begin{array}{c}
\begin{array}{ccccc}
SU(12)_C & \times & SU(2)_L & \times & SU(2)_R \\
12 & & 2 & & 1 \\
\overline{12} & & 1 & & 2
\end{array} \\
\downarrow \\
SU(12)_C \times SU(2)_L \times SU(2)_R \\
\downarrow \\
SU(8)_C \times SU(4)_{C1} \times SU(2)_L \times SU(2)_R \times U(1) \\
\downarrow \\
SU(4)_{C3} \times SU(4)_{C2} \times SU(4)_{C1} \times SU(2)_L \times SU(2)_R \times U(1) \times U(1) \\
\downarrow \\
\downarrow \\
SU(4)_C \times SU(2)_L \times SU(2)_R \\
\downarrow \\
\downarrow \\
SU(3)_C \times SU(2)_L \times U(1)_Y \\
\\
\underbrace{three\ quark\ and\ lepton\ families}
\end{array}
$$

Therefore, there are three generations of quarks and leptons under the standard group of $SU(3)_C \times SU(2)_L \times U(1)_Y$. However, before taking this model seriously, more works have to be done.

In sum, we have found that the requirements of minimality and freedom from all three chiral gauge anomalies in four dimensions lead to a unique set of Weyl representations of the standard group, corresponding to the observed quarks and leptons of one family. Furthermore, the $U(1)_Y$ charges of

these quarks and leptons are quantized and correctly determined by adding the mixed anomaly-free condition and thus a long-standing puzzle of the electric charge quantization of quark and lepton can be solved within the content of the standard model. The determination of the uniqueness of the standard model due to the anomaly cancellations argues strongly for new physics beyond the standard model, especially some form of the quark-lepton unification. However, there is still no answer to the family problem. Maybe there are possibly some as-yet-unidentified anomalies in four dimensions, or larger symmetries like $SU(12)_C \times SU(2)_L \times SU(2)_R$, or higher dimensions,[13] or presons,[15] or others.

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1. Glashow, S. L., Nucl. Phys. **22**, 579 (1961); Weinberg, S., Phys. Rev. Lett. **19**, 1264 (1964); Salam, A., " Elementary Particle Theory", ed. Svarthom, N. (Almquist and Wilsell, Stockholm, 1968).
2. Adler, S. Phys. Rev. **117**, 2426 (1969); Bell, J and Jackiw, R., Nuovo Cimento **51A**, 47 (1969); Bardeen, W., Phys. Rev. **184**, 1848 (1969).
3. Witten, E., Phys. Lett. **B117**, 324 (1982).
4. Delbourgo, R. and Salam, A., Phys. Lett. **B40**, 381 (1972); see also Eguchi, T. and Frend, T., Phys. Rev. Lett. **37**, 1251 (1976).
5. Alvarez-Guamé, L. and Witten, E., Nucl. Phys. **B234**, 269 (1983).
6. Georgi, H. and Glashow, S. L. Phys. Rev. **D6**, 1159 (1972); Okubo, S., Phys. Rev. **D16**, 3528 (1977).
7. Atiyah, M. T., Proc. of the *XVIIIth Solvay Conference on Physics* at Univ. of Texas, Austin, Texas, 1982 (Phys. Rep. **104**, 203 (1984)).
8. Geng, C. Q., Marshak, R. E., Zhao, Z. Y., and Okubo, S., Phys. Rev. **D36**, 1953 (1987).
9. Bouchiat, C., Illiopoulos, J. and Meyer, Ph., Phys. Lett. **B73**, 519 (1972); Gross, D. J. and Jackiw, R., Phys. Rev. **D6**, 477 (1972).
10. For reviews, see Glashow, S. L., "Quarks and Leptons", ed. Levy, M., (Cargese, 1979); "Fundamental Interactions", ed. Levy, M., (Cargese, 1981).
11. Geng, C. Q. and Marshak, R. E., Phys. Rev. **D39**, 693 (1989).
12. Geng, C. Q., Phys. Rev. **D39**, 2402 (1989).
13. For a review, see Peccei, R. D., DESY Report No. DESY-88-078.
14. Pati, J. C. and Salam, A, Phys. Rev. **D10**, 275 (1974).
15. Geng, C. Q. and Marshak, R. E., Phys. Rev. **D35**, 2278 (1987).